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TOPOLOGICAL ANALYSIS OF SUPER-DYADIC GRAPHS

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Abstract. For the topological analysis of complex networks having a super-dyadic nature, hypergraph representation is more convenient than graph representation in many cases. This paper presents new parameters that can be used to investigate the interactions between various network nodes. Based on this parameter, topological indices are defined and discussed. To construct a hypergraph with a highly modular structure, this paper introduces an inequality.

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1. Introduction

Numerous problems in the real world cannot be represented by a simple graph; in a simple graph, each edge joins exactly two nodes. A unique kind of graph where one edge can link two or more nodes is called a hypergraph. A hypergraph Z is represented mathematically by the pair (Y, E), where Y is the vertex set and E is a collection of subsets of Y (hyperedge set)[15, 21].

Graphs only go so far in simulating the complexity of many real-world problems. More generally than graphs, hypergraphs can model intricate systems. Gene interactions, risk management, computer networks, social networks, and visual classification are just some of the many applications for hypergraphs. We can display information regarding relationships that involve more than one object due to the use of hypergraphs [7].

Consider, for instance, a graph that shows the authors of papers and their collaborations on papers. Imagine that you have three authors as well as the following graph, in which the authors serve as the vertices of the network and the edges represent the authors' contributions to a publication. As a result of this, we are aware that each author collaborated with two of the other authors at

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some point on a paper; however, we are unable to identify whether this occurred independently on each of the three papers or whether it occurred simultaneously on a single manuscript. On the other hand, we can illustrate this using a hypergraph, with each edge standing for a different collaboration that was included in the publication.

Because of its versatility in representing group relationships, hypergraphs are often used to address complex technical issues [1, 21]. Social media networks such as Facebook and LinkedIn are concrete illustrations of hypergraphs in the real world. Each user is a vertex that can belong to a group and a hyperedge is a group.

Topological indices are numerical values that represent a graph's structural characteristics. Chemical graph theory, a field of mathematics that combines graph theory and chemistry, showed a great deal of interest in it. Some topological indices have shown to be helpful in the process of identifying quantitative structure-property relationships (QSPRs) and quantitative structure-activity relationships (QSARs) [11, 22]. Numerous topological indices are derived from various parameters, including degree, eccentricity, distance, and so on [3, 9, 16]. In the literature on mathematical chemistry and mathematics, several degree-based graph invariants are studied, but Zagreb indices are among the most prominent.

First Zagreb index of a graph Q is defined as $Z_1(Q) = \sum_{uv \in E(Q)}[d(u) + d(v)] = \sum_{v \in V(Q)} d^2(v)$ [5, 18, 19]. Topological indices are significant numbers that represent different aspects of the network's connection.

Many science-related networks, such as social networks, computer networks, and metabolic and regulatory networks, naturally divide into communities or modules. Detecting and characterizing this modular structure is one of the outstanding difficulties in networked system research. A connected element of a graph is generalized into a module. One module, however, can be a proper subset of another, unlike connected components. Hence, modules result in a recursive (hierarchical) breakdown of the network as opposed to merely a division. Techniques for identifying modules, or communities inside networks, are particularly interesting as they reveal significant subnetworks or building blocks that are especially closely coupled, frequently correlating to specialized functional components [13, 17]. This work is focused on the construction of modular structure for hypergraph.

This paper introduces a new parameter, hyperedge degree $d_h(l)$, and new topological indices based on this parameter value. This parameter depends on the degree of each vertex of the hyperedge l. A social network like Facebook or LinkedIn can be represented as a hypergraph, where users are the vertices and groups are the hyperedges. Hence, a high $d_h(l)$ indicates that group l has a significant impact on other groups. If users in group l are also members of several other groups, then $d_h(l)$ will be high. This attribute might be very useful for disseminating crucial information (emergency situations) through social media. In this case, this message can be distributed to some groups with higher $d_h(l)$.

So, in a short period of time and with little work, it will spread and produce significant benefits.

The representation of networks as hypergraphs is the main topic of this research. Using these hypergraph implementations in social networks like Facebook, it is possible to discover users with more influence over others. The first section is used to introduce the work carried out. In this work, some topological indices are defined based on the new parameter introduced in the second section. Graph operations are covered in the third section, which can be used to construct larger networks from smaller ones. The fourth section deals with the construction of a modular structure.

2. Hypergraph Topological Indices

In this section, new topological indices based on the hypergraph degrees of certain graph families are defined and discussed. Some graph families Q that facilitate small-world organization are the wheel, firefly, windmill, and so on [14]. Windmill graph W_p^b is an undirected graph made by combining b copies of the entire graph K_p at a common universal vertex for $p(>2)$ and $b(>2)$, and wheel graph W_f is a graph of f vertices that is generated by joining every cycle vertex to one universal vertex, firefly graph $F_{c,d,t}$ is a graph composed of c triangles, t pendant paths of length 2, and d pendant edges sharing a common vertex [4, 10, 12]. For our convenience, each set of vertices that induces a complete graph is treated as a hyperedge. A complete graph is treated as a hypergraph with only one hyperedge. The impact of a hyperedge's peripheral connections with other hyperedges on $d_h(l)$ values is explained in this section.

Definition 2.1. If Y and E are the sets of vertex and hyperedge, respectively, and each hyperedge is a non-empty subset of Y, then the pair (Y, E) is called a hypergraph Z.

Definition 2.2. Let Z be a hypergraph with vertex set $Y = \{y_1, y_2, ..., y_n\}$ and let l and y be a hyperedge and vertex respectively. Then hyperedge degree is defined as $d_h(l) = \sum_{y \in l} d_h(y) - |l|$ where $d_h(y)$ is the total number of vertex y contained hyperedges in Z and $|l|$ is the number of vertices in l.

Definition 2.3. Let Z be a hypergraph with edge set E. Then hyper first Zagreb index and hyper degree index are defined as, $HZ_1(Z) = \sum_{l \in E} d_h^2(l)$ and $HD(Z) = \sum_{l \in E} d_h(l)$ respectively.

Lemma 2.4. *Let l be a hyperedge and each hyperedge is a set of vertices induce a complete graph and let $n(E)$ be the total number of hyperedges in Z.*

- *If C_f is a cycle graph(f vertices). Then $n(E) = f$ and $d_h(l) = 2\ \forall l \in C_f$. So, $HD(C_f) = 2f$ and $HZ_1(C_f) = 4f$.*
- *For a tree T, $d_h(l) = n(v) + n(u) - 2\ \forall l \in T$, where $v, u \in l$, $v \neq u$ and $n(v)$ is a neighbor set of vertex v. Then $HD(T) = \sum_{vu \in E(T)} (n(v) + n(u) - 2)$ and $HZ_1(T) = \sum_{vu \in E(T)} (n(v) + n(u) - 2)^2$.*
  In particular,

– *If P_f is a path(f vertices), then*
$$d_h(l) = \begin{cases} 1 & \textit{;if } l \textit{ is set of vertices induces an end edge} \\ 2 & \textit{; otherwise} \end{cases}.$$
Therefore $HD(P_f) = 2(f-2)$ and $HZ_1(P_f) = 4(f-3)+2$.

– *For a star graph S_t with $t+1$ vertices, $d_h(l) = t-1$, $HD(S_t) = t(t-1)$ and $HZ_1(S_t) = t(t-1)^2$.*

Proof. In the case of C_f and tree T, sets of vertices that induce K_2 are the hyperedges. So, $HD(C_f) = 2f$, $HZ_1(C_f) = 4f$, $HD(T) = \sum_{uv\in E(T)}(n(v)+n(u)-2)$ and $HZ_1(T) = \sum_{uv\in E(T)}(n(v)+n(u)-2)^2$. □

Lemma 2.5. *For a Windmill graph W_p^b, $n(E) = b$ and $d_h(l) = b-1$ $\forall l \in W_p^b$ where total number of hyperedges in W_p^b is denoted by $n(E)$.*

Proof. Since a windmill graph is a graph consisting of b number of K_p and each set of vertices induces a complete graph is a hyperedge, $n(E) = b$. So,
$$d_h(y) = \begin{cases} b & \text{;if } y \text{ is the center} \\ 1 & \text{; otherwise} \end{cases} \text{ and hence}$$
$d_h(l) = \sum_{y\in l} d_h(y) - |l| = b+p-p-1 = b-1$ □

Theorem 2.6. *For a Windmill graph W_p^b, $HD(W_p^b) = (b-1)b$ and $HZ_1(W_p^b) = (b-1)^2 b$.*

Proof. $d_h(l) = b-1$ $\forall l \in Q$ (from lemma(2.2)). So, $HD(W_p^b) = \sum_{l\in E} d_h(l) = (b-1)\times(b-1)...\times(b-1)$ (b times) $= (b-1)b$ and $HZ_1(W_p^b) = \sum_{l\in E} d_h^2(l) = (b-1)^2\times(b-1)^2\times...\times(b-1)^2$ (b times) $= (b-1)^2 b$. □

Lemma 2.7. *For a Firefly graph $F_{c,d}$ (with $t = 0$), $n(E) = c+d$ and $d_h(l) = c+d-1$ $\forall l \in F_{c,d}$.*

Proof. Since $F_{c,d}$ contains c set of vertices that induces triangles (means K_3) and d set of vertices induces pendent edges (means K_2) and each set of vertices induces complete graph is a hyperedge, the total number of hyperedges in $F_{c,d}$ is $c+d$. So,
$$d_h(y) = \begin{cases} c+d & \text{;if } y \text{ is the center} \\ 1 & \text{; otherwise} \end{cases} \text{ and hence}$$
$$\begin{aligned} d_h(l) &= \sum_{y\in l} d_h(y) - |l| \\ &= \begin{cases} 1+1+(c+d)-3 & \text{;if } l \text{ is set of vertices induces } K_3 \\ 1+c+d-2 & \text{; if } l \text{ is set of vertices induces } K_2 \end{cases} \\ &= \begin{cases} c+d-1 & \text{;if } l \text{ is set of vertices induces } K_3 \\ c+d-1 & \text{; if } l \text{ is set of vertices induces } K_2 \end{cases} \end{aligned}$$
□

Theorem 2.8. *If Q be $F_{c,d}$ then $HD(Q) = (c+d-1)(c+d)$ and $HZ_1(Q) = (c+d-1)^2(c+d)$.*

Proof. $d_h(l) = c+d-1 \ \forall l \in Q$ (from lemma(2.7)). So, $HD(Q) = \sum_{l\in E} d_h(l) = (c+d-1)\times(c+d-1)... \times(c+d-1)$ (c+d times) $= (c+d-1)(c+d)$ and $HZ_1(Q) = \sum_{l\in E} d_h^2(l) = (c+d-1)^2 \times (c+d-1)^2 ... \times (c+d-1)^2$ (c+d times) $= (c+d-1)^2(c+d)$. □

Lemma 2.9. *For a Wheel graph W_f (f vertices), $n(E) = f$ and $d_h(l) = 1+f$ $\forall l \in W_f$ where total number of hyperedges in W_f is denoted by $n(E)$.*

Proof. Since the wheel graph contains f set of vertices that induces triangles (means K_3) and each set of vertices induces a complete graph is a hyperedge, the total number of hyperedges in W_f is f. So,

$d_h(y) = \begin{cases} f & \text{;if } y \text{ is the center} \\ 2 & \text{; otherwise} \end{cases}$ and hence

$d_h(l) = \sum_{y\in l} d_h(y) - |l| = 2+2+f-3 = f+1$ □

Theorem 2.10. *If $Q \cong W_f$ then $HD(Q) = f(f+1)$ and $HZ_1(Q) = f(f+1)^2$.*

Proof. $d_h(l) = f+1 \ \forall l \in Q$ (from lemma(2.9)). So, $HD(Q) = \sum_{l\in E} d_h(l) = (f+1)\times(f+1)\times ... \times(f+1)$ (f times) $= f(f+1)$ and $HZ_1(Q) = \sum_{l\in E} d_h^2(l) = (f+1)^2 \times (f+1)^2 \times ... \times (f+1)^2$ (f times) $= f(f+1)^2$. □

3. Hypergraph Topological Indices and Graph Operations

Graph operations enable us to build big networks out of smaller ones and vice versa. Graph operations cartesian product, join, composition, and corona products are defined as, the cartesian product $Q_1 \times Q_2$ of graphs Q_1 and Q_2 is a graph with vertex set $V(Q_1 \times Q_2) = V(Q_1) \times V(Q_2)$ and $(c, x)(d, y)$ is an edge of $Q_1 \times H_2$ if $c = d$ and $xy \in Q_2$, or $cd \in E(Q_1)$ and $x = y$; the join $Q_1 + Q_2$ of graphs Q_1 and Q_2 is a graph with vertex set $V(Q_1) \cup V(Q_2)$ and edge set $E(Q_1) \cup E(Q_2) \cup \{xy; x \in V(Q_1) \text{ and } y \in V(Q_2)\}$; the composition $Q_1 \circ Q_2$ of graphs Q_1 and Q_2 with disjoint vertex sets $V(Q_1)$ and $V(Q_2)$ and edge sets $E(Q_1)$ and $E(Q_2)$ is the graph with vertex set $V(Q_1) \times V(Q_2)$ and $x = (x_1, y_1)$ is adjacent to $y = (x_2, y_2)$ whenever x_1 is adjacent to x_2 or $x_1 = x_2$ and y_1 is adjacent to y_2; the corona product $Q_1 + Q_2$ of graphs Q_1 and Q_2 is a graph formed by taking one copy of Q_1 and $|V(Q_1)|$ copies of Q_2, and then connecting via an edge, with each vertex of the kth copy of Q_2 labeled (Q_2, k) with the kth vertex of Q_1 [8, 20].

These graph operations—join, composition, cartesian, and corona products, among others—allow one to start from a collection of smaller communities and produce a large community or entire network, and vice versa. This section explains a number of graph operations that help build hypergraphs and talks about the outcomes of such operations on hypergraphs. This section specifically addresses the outcomes of graph operations on hypergraphs.

Lemma 3.1. *Let $Z_1 = K_r$ and $Z_2 = K_s$. Then cartesian product $Z = Z_1 \times Z_2$ is a hypergraph with edge set $E(Z) = \{E(Z_1)(s \text{ times}), E(Z_2)(r \text{ times})\}$ and vertex set $Y(Z) = Y(Z_1) \times Y(Z_2)$.*

Proof. From the definition of the cartesian product of graphs and hypergraphs. □

Theorem 3.2. *Let $Z_1 = K_r$ and $Z_2 = K_s$. Then the cartesian product $Z = Z_1 \times Z_2$ contains $r+s$ hyperedges and* $d_h(l) = \begin{cases} r & \text{;if } l \text{ is set of vertices induces } K_r \\ s & \text{;if } l \text{ is set of vertices induces } K_s \end{cases}$ *and $HD(Z) = 2|Y(Z_2)||Y(Z_1)|$ and $HZ_1(Z) = (|Y(Z_2)|+|Y(Z_1)|)|Y(Z_2)||Y(Z_1)|$*

Proof. From lemma(3.1), clear that Z contains $r+s$ hyperedges. There are two types of edges ϵ_1 and ϵ_2 where ϵ_1 is a set of vertices that induces K_r and ϵ_2 is a set of vertices that induces K_s. Here $d_h(y) = 2\ \forall y \in Z$ and $E(Z) = \{Y(K_s), ..., Y(K_s)(r \text{ times}), Y(K_r), ..., Y(K_r)(s \text{ times})\}$ where $Y(Z)$ denotes the vertex set of Z. Since $d_h(l) = \sum_{y\in l} d_h(y) - |l|$, $d_h(\epsilon_2) = 2+2+...+2(s$ times$) - s = s$ and $d_h(\epsilon_1) = 2+2+...+2(r$ times$) - r = r$. So,

$$d_h(l) = \begin{cases} s & \text{;if } l \text{ is set of vertices induces } K_s \\ r & \text{;if } l \text{ is set of vertices induces } K_r \end{cases}$$

$$\begin{aligned} HD(Z) &= \textstyle\sum_{l\in Z_1\times Z_2} d_h(l) \\ &= \textstyle\sum_{l\in Z_1} d_h(l) + \sum_{l\in Z_2} d_h(l) \\ &= (2r-r)\times s + (2s-s)\times r \\ &= 2rs \\ &= 2|Y(Z_2)||Y(Z_1)| \end{aligned}$$

$$\begin{aligned} HZ_1(Z) &= \textstyle\sum_{l\in Z_1\times Z_2} d_h^2(l) \\ &= \textstyle\sum_{l\in Z_1} d_h^2(l) + \sum_{l\in Z_2} d_h^2(l) \\ &= (2r-r)^2\times s + (2s-s)^2\times r \\ &= rs(r+s) \\ &= |Y(Z_1)||Y(Z_2)|(|Y(Z_1)|+|Y(Z_2)|) \end{aligned}$$

□

Theorem 3.3. *If Z_1 and Z_2 are complete graphs, then the composition of Z_1 and Z_2 is also complete.*

Proof. Let K_r and K_s be complete graph with vertex sets $\{v_1, v_2, ..., v_s\}$ and $\{u_1, u_2, ..., u_r\}$ respectively. Since edge sets $E(K_r) = \{u_1u_2, ..., u_1u_n, u_2u_3, ..., u_2u_n, ..., u_{r-1}u_r\}$ and $E(K_s) = \{v_1v_2, ..., v_1v_s, v_2v_3, ..., v_2v_s, ...v_{s-1}v_s\}$ and vertex set $Y(K_r \circ K_s) = Y(K_r) \times Y(K_s) = \{u_iv_j;\ j = 1, 2, ..., s \text{ and } i = 1, 2, ..., r\}$, all edges of K_{rs} except $\{(u_iv_j)(u_iv_k)\}$ are covered by the first condition of composition, where $j \neq k$, $j, k = 1, 2, ..., s$ and $i = 1, 2, ..., r$. These remaining edges for the completion of the complete graph are covered by the second requirement of composition. □

Lemma 3.4. *Join product Z of hypergraphs Z_1 and Z_2 is a hypergraph (Z_1+Z_2) with hyperedge set $E(Z) = \{e' \cup e^*; \forall e' \in E(Z_1) \text{ and } e^* \in E(Z_2)\}$ and vertex set $Y(Z) = Y(Z_1) \cup Y(Z_2)$.*

Proof. From join product definition of graphs and hypergraph definition. □

Theorem 3.5. *Let $\epsilon = e' \cup e^*$ be a hyperedge of Z where $Z = Z_1 + Z_2$ is the join of hypergraphs Z_1 and Z_2, then Z contains r_1r_2 hyperedges where r_1 and r_2 are the number of hyperedges in Z_1 and Z_2 respectively and $d_h(l) = r_2(d_h(e') + |e'|) + r_1(d_h(e^* + |e^*|)$, where $e' \in E(Z_1)$ and $e^* \in E(Z_2)$ and $HD(Z) = n_2^2 HD(Z_1) + r_1^2 HD(Z_2) + r_2(r_2-1)\sum_{e'}|e'| + r_1(r_1-1)\sum_{e^*}|e^*|$.*

Proof. Let Z_1 contains r_1 hyperedges and Z contains r_2 hyperedges then $d_h(Y) = \begin{cases} r_1 d_h(y) & ;\text{ if } y \in Y(Z_2) \\ r_2 d_h(y) & ;\text{ if } y \in Y(Z_1) \end{cases}$ and number of hyperedges in Z, $n(E(Z)) = n(E(Z_1 + Z_2)) = n(E(Z_1)) \times n(E(Z_2)) = r_1 r_2$.

Let $e'_1, e'_2, ..., e'_{r_1}$ are hyperedges of Z_1 and $e^*_1, e^*_2, ..., e^*_{r_2}$ are hyperedges of Z_2, then
$E(Z) = E(Z_1+Z_2) = \{(e'_1 \cup e^*_1), (e'_1 \cup e^*_2), ..., (e'_1 \cup e^*_{r_2}), (e'_2 \cup e^*_1), (e'_2 \cup e^*_2), ..., (e'_2 \cup e^*_{r_2}), ..., (e'_{r_1} \cup e^*_1), (e'_{r_1} \cup e^*_2), ..., (e'_{r_1} \cup e^*_{r_2})$. Let $e' \in E(Z_1)$ and $e^* \in E(Z_2)$ then

$$\begin{aligned} d_{h_{Z_1+Z_2}}(l) &= d_h(e^* + e'); e^* \in Z_2, e' \in Z_1 \\ &= \textstyle\sum_{y \in Y(e^*+e')} d_h(y) - |e^* + e'| \\ &= r_1 \textstyle\sum_{y^* \in Y(e^*)} d_h(y^*) + r_2 \sum_{y \in Y(e')} d_h(y) - |e'| - |e^*| \\ &= r_2 d_h(e') + r_1 d_h(e^*) + (r_2-1)|e'| + (r_1-1)|e^*| \end{aligned}$$

$$\begin{aligned} HD(Z_1+Z_2) &= \textstyle\sum_{l \in E(Z_1+Z_2)} d_h(l) \\ &= \textstyle\sum_{e' \in E(Z_1), e^* \in E(Z_2)} d_h(e' \cup e^*) \\ &= r_2(d_h(e'_1) + |e'_1|) + r_1(d_h(e^*_1) + |e^*_1| - (|e'_1| + |e^*_1|)) + \\ &\quad r_2(d_h(e'_1) + |e'_1|) + r_1(d_h(e^*_2) + |e^*_2|) - (|e'_1| + |e^*_2|) + ... + \\ &\quad r_2(d_h(e'_1) + |e'_1|) + r_1(d_h(e^*_{r_2}) + |e^*_{r_2}|) - (|e'_1| + |e^*_{r_2}|) + \\ &\quad r_2(d_h(e'_2) + |e'_2|) + r_1(d_h(e^*_1) + |e^*_1|) - (|e'_2| + |e^*_1|) \\ &\quad + r_2(d_h(e'_2) + |e'_2|) + r_1(d_h(e^*_2) + |e^*_2|) - (|e'_2| + |e^*_2|) + ... + \\ &\quad r_2(d_h(e'_2) + |e'_2|) + r_1(d_h(e^*_{r_2}) + |e^*_{r_2}|) - (|e'_2| + |e^*_{r_2}|) + ... + \\ &\quad r_2(d_h(e'_{r_1}) + |e'_{r_1}|) + r_1(d_h(e^*_1) + |e^*_1|) - (|e'_{r_1}| + |e^*_1|) + \\ &\quad r_2(d_h(e'_{r_1}) + |e'_{r_1}|) + r_1(d_h(e^*_2) + |e^*_2|) - (|e'_{r_1}| + |e^*_2|) \\ &\quad + ... + r_2(d_h(e'_{r_1}) + |e'_{r_1}|) + r_1(d_h(e^*_{r_2}) + |e^*_{r_2}|) - (|e'_{r_1}| + |e^*_{r_2}|) \\ &= r_2^2 \textstyle\sum_{i=1}^{r_1}(d_h(e'_i) + |e'_i|) + r_1^2 \sum_{j=1}^{r_2}(d_h(e^*_j) + |e^*_j|) - \\ &\quad (r_2 \textstyle\sum_{i=1}^{r_1} |e'_i| + r_1 \sum_{j=1}^{r_2} |e^*_j|) \\ &= r_2^2 \textstyle\sum_{e' \in E(Z_1)}(d_h(e') + |e'|) + r_1^2 \sum_{e^* \in E(Z_2)}(d_h(e^*) + |e^*|) \\ &\quad - (r_2 \textstyle\sum_{e' \in E(Z_1)} |e'| + r_1 \sum_{e^* \in E(Z_2)} |e^*|) \\ &= r_2^2 HD(Z_1) + r_1^2 HD(Z_2) + r_2(r_2-1) \textstyle\sum_{e'} |e'| + \\ &\quad r_1(r_1-1) \textstyle\sum_{e^*} |e^*| \end{aligned}$$

□

Lemma 3.6. *Let $Z = Z_1 \odot Z_2$ be the corona product of Z_1 and Z_2 where $Z_1 = S_t$ and $Z_2 = K_f$, then Z is a hypergraph with edge set $E(Z) = \{Y(K_{f+1})((t+1)$ times$), Y(K_2)(t$ times$)\}$ and $|Y(Z)| = (f+1)(t+1)$.*

Proof. From corona product definition of graphs and hypergraph definition. □

Theorem 3.7. *Let $Z = Z_1 \odot Z_2$ be the corona product of $Z_1 = S_t$ and $Z_2 = K_f$. Then Z contains $2t+1$ hyperedges and*

$$d_h(l) = \begin{cases} t & \textit{;if } l \textit{ is the set of vertices induces } K_{f+1} \textit{ attached to the center} \\ t+1 & \textit{;if } l \textit{ is set of vertices induces } K_2 \textit{ (the pendant edge of } S_t\textit{)} \\ 1 & \textit{;otherwise} \end{cases}$$

and $HD(Z) = HD(Z_1) + 4t$ *and* $HZ_1(Z) = HZ_1(Z_1) + 5t^2 + t$

Proof. From lemma(3.6), clear that $n(E) = 2t + 1$. There are three types of edge ϵ_1, ϵ_2, ϵ_3. ϵ_1 is set of vertices induces K_2, ϵ_2 is set of vertices induces K_{f+1} attached to the center, ϵ_3 is set of vertices induces K_{f+1} which is not attached to the center. Here $E(Z) = \{Y(K_{f+1})((t+1) \text{ times}), Y(K_2)(t \text{ times})\}$,

$$d_h(y) = \begin{cases} 2 & \text{;if } y \text{ is the pendent vertex of } S_t \\ t+1 & \text{;if } y \text{ is the center of } S_t \\ 1 & \text{;otherwise} \end{cases}$$

. Since $d_h(l) = \sum_{y \in l} d_h(y) - |l|$, $d_h(\epsilon_3) = 2+1+1+...+1(f \text{ times}) - (f+1) = 1$, $d_h(\epsilon_2) = (t+1)+1+1+...+1(f \text{ times}) - (f+1) = t$ and $d_h(\epsilon_1) = t+1+2-2 = t+1$. So,

$$d_h(l) = \begin{cases} t+1 & \text{;if } l \text{ is set of vertices induces } K_2 \text{ (the pendant edge of } S_t\text{)} \\ t & \text{;if } l \text{ is the set of vertices induces } K_{f+1} \text{ attached to the center} \\ 1 & \text{;otherwise} \end{cases}$$

$HD(Z) = \sum_{l \in Z} d_h(l) = t \times (t+1) + 1 \times t + t \times 1 = t^2 + 3t = t(t-1) + 4t = HD(Z_1) + 4t$

$HZ_1(Z) = \sum_{l \in Z} d_h^2(l) = t \times (t+1)^2 + 1 \times t^2 + t \times 1^2 = t(t-1)^2 + 5t^2 + t = HZ_1(Z_1) + 5t^2 + t$

□

Lemma 3.8. *Let* $Z = Z_1 \odot Z_2$ *be the corona product of* Z_1 *and* Z_2 *where* $Z_1 = K_r$ *and* $Z_2 = K_s$*. Then* Z *is a hypergraph with edge set* $E(Z) = \{Y(K_{s+1})(r$ *times*$), Y(K_r)\}$ *and* $|Y(Z)| = r(s+1)$*.*

Proof. From corona product definition of graphs and hypergraph definition. □

Theorem 3.9. *Let* $Z_1 = K_r$ *and* $Z_2 = K_s$*, then the corona product* $Z = Z_1 \odot Z_2$ *contains* $r+1$ *hyperedges and* $d_h(l) = \begin{cases} 1 & \textit{;if } l \textit{ is set of vertices induces } K_{s+1} \\ r & \textit{;if } l \textit{ is set of vertices induces } K_r \end{cases}$ *and* $HD(Z) = 2n(Z_1)$ *and* $HZ_1(Z) = n(Z_1)[n(Z_1) + 1]$ *where* $n(Z)$ *is the cardinality of vertex set of* Z*.*

Proof. From lemma(3.8), clear that Z contains $r+1$ hyperedges. There are two types of edge ϵ_1 and ϵ_2. ϵ_1 is set of vertices induces K_n and ϵ_2 is set of vertices induces K_{s+1}. Here $E(Z) = \{Y(K_{s+1})(r \text{ times}), Y(K_r)\}$,

$d_h(y) = \begin{cases} 2 & \text{;if } y \in Y(K_r) \\ 1 & \text{;if } y \in Y(K_s) \end{cases}$ and $d_h(l) = \sum_{y \in l} d_h(y) - |l|$. Therefore $d_h(\epsilon_2) = 2+1+1+...+1(s \text{ times}) - (1+s) = 1$ and $d_h(\epsilon_1) = 2+2+...+2(r \text{ times}) - r = r$. So,

$$d_h(l) = \begin{cases} 1 & \text{;if } l \text{ is set of vertices induces } K_{s+1} \\ r & \text{;if } l \text{ is set of vertices induces } K_r \end{cases}$$
$HD(Z) = \sum_{l \in Z} d_h(l) = r \times 1 + 1 \times r = 2r = 2n(Z_1)$
$HZ_1(Z) = \sum_{l \in Z} d_h^2(l) = r \times 1^2 + 1 \times^2 = r + r^2 = n(Z_1)[n(Z_1) + 1]$ □

4. Modular Structure Construction

In the context of networks, complicated interactions between several nodes can be represented as hypergraphs. There are more connections between nodes within modules in highly modular networks than there are between modules. A structural measure known as a network's modularity assesses how easily a network may be divided into more manageable sub-networks, sometimes known as groups, communities, or clusters. More intra-module connections and fewer inter-module connections are indicative of higher modularity[2, 6].

Let Z be the hypergraph and let l be a hyperedge in Z. The hyperedge degree is hence $d_h(l) = \sum_{y \in l} d_h(y) - |l|$, where $d_h(y)$ is the total number of vertex y contained hyperedges in Z and $|l|$ is the number of vertices in l. For each hyperedge l in Z, the clique strength of internal connections is approximately equal to $x(x-1)/2$, where x is the edge's size.

The community's weak external links are just as significant as its strong internal connections. When x is the size of edge l, then a strongly linked local region satisfies $x(x-1)/2 \geq \sum_{y \in l} d_h(y) - x$.
i.e., $|l|(|l|-1)/2 \geq 2 \sum_{y \in l} d_h(y) - |l|$
$\Rightarrow |l|^2 - |l| \geq \sum_{y \in l} d_h(y) - |l|$
$\Rightarrow (|l|^2 + |l|)/2 \geq \sum_{y \in l} d_h(y)$

So, this inequality helps to construct the modular structure of a hypergraph. Each module in a modular system is framed by dense connections inside each module and weak connections across modules. This inequality is satisfied by a hypergraph representation that results in a highly modular structure. The network can be effectively organized so that there are dense connections inside the group and sparse connections outside by optimizing this inequality. To apply this, edges should be created with nearly cliques.

The efficient development of modular structures for hypergraphs is the subject of discussion in this section. This section highlights the significance of this modular building structure.

Theorem 4.1. *If the hyperedges of the hypergraph $S_t \odot K_f$ satisfy the inequality for all f and t, then the hypergraph can have a highly modular structure.*

Proof. Theorem (3.7) mentions three different kinds of hyperedges for $S_t \odot K_f$ by classifying vertex sets that produce complete graphs as hyperedges. The first set of vertices, ϵ_1, induces an edge with vertex degrees $t+1$ and 2; the second set, ϵ_2, induces K_{f+1} combined to the pendent vertex of S_t with vertex degrees 2 and 1 (for f vertices); the third set, ϵ_3, induces K_{f+1} combined to center vertex of S_t

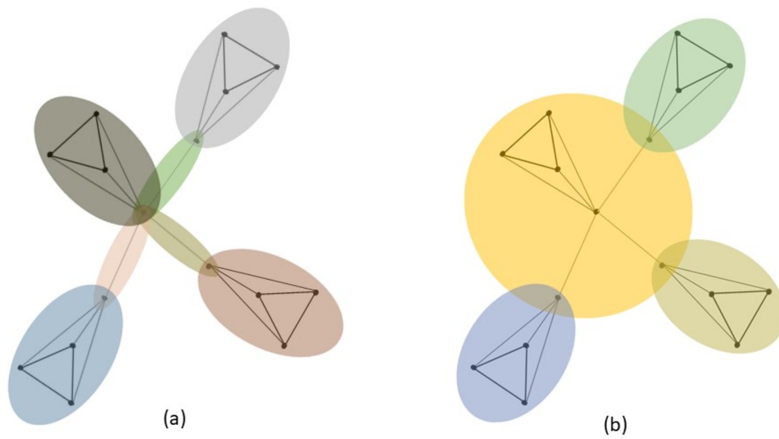

FIGURE 1. (a) Modular structure of $S_3 \odot K_3$ hypergraph as set of vertices induces complete graph as hyperedge; (b) Highly modular structure of $S_3 \odot K_3$ hypergraph.

with vertex degrees $t+1$ and 1 (for f vertices)(Fig. 1(a)).). However, the external connections within this group outweigh the internal ones. Thus, in this instance, the modular structure's modularity will be reduced. The efficacy of hyperedge selection can be verified by using the inequality $(|l|^2 + |l|)/2 \geq \sum_{y \in l} d_h(y)$.

- For ϵ_1, $t + 1 + 2 \leq (2^2 + 2)/2 \Rightarrow t \leq -3/2$.
- For ϵ_2, $f + 2 \leq ((f+1)^2 + (f+1))/2 \Rightarrow 2 \leq f(f+1)$.
- For ϵ_3, $f + t + 1 \leq ((f+1)^2 + (f+1))/2 \Rightarrow 2t \leq f(f+1)$

Given that $t \geq 1$, there is a contradiction in the case of ϵ_1. To put it another way, because it is a grouping with less modularity, the inequality is not satisfied. Furthermore, under t and f constraints, some hyperedges, in this case, satisfy the inequality. We now need to regroup them to enhance the modular framework.

(1) One ϵ_1 type edge combined with one ϵ_2 type edge, then $f + 1 + t + 1 \leq ((f+2)^2 + (f+2))/2 \Rightarrow 2t \leq (f+1)(f+2)$
(2) One ϵ_1 type edge combined with ϵ_3 type edge, then $f + 1 + t + 1 \leq ((f+2)^2 + (f+2))/2 \Rightarrow 2t \leq (f+1)(f+2)$
(3) Two ϵ_1 type edge combined with ϵ_3 type edge, then $f + t - 1 + 2 < ((f+3)^2 + (f+3))/2 \Rightarrow 2t \leq f^2 + 5f + 10$
(4) All ϵ_1 type edge combined with ϵ_3 type edge, then $f + 1 + 2t \leq ((f + 1 + t)^2 + (f + 1 + t))/2 \Rightarrow t \leq f^2 + t^2 + f(2t+1)$

Thus, (4) is the best regrouping since it has two types of hyperedges (all edges of type ϵ_1 with ϵ_3 and ϵ_2) and very few outside connections relative to inner connections. Stated otherwise, regrouping (4) satisfies all f and t and produces the lowest value for $d_h(y)$ when compared to alternative regroupings.

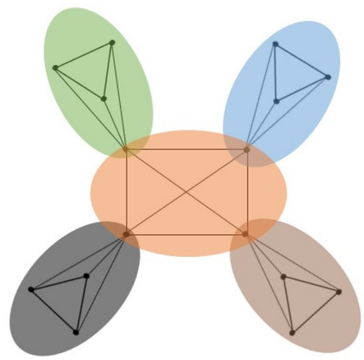

FIGURE 2. Modular structure of $K_4 \odot K_3$ hypergraph.

Modular structures with high modularity hence satisfy the inequality $(|l|^2 + |l|)/2 \geq \sum_{y \in l} d_h(y)$ for all f and t. □

Theorem 4.2. *When $r > 2$, hyperedges can result in complete graphs generated by modular structures of $K_r \odot K_s$ with vertex sets.*

Proof. Theorem 3.9 presents two varieties of hyperedges. A collection of vertices in the first type, ϵ_1, induces K_r with vertex degree 2 for all vertices, while a set of vertices in the second type, ϵ_2, induces K_{s+1} with vertex degrees 2 and 1 (for s vertices) (Fig. 2). It is now possible to evaluate how effective hyperedge selection is.

(1) For ϵ_1, $2r \leq (r^2 + r)/2 \Rightarrow 0 \leq r(r-3) \Rightarrow r > 2$
(2) For ϵ_2, $2 + s \leq ((1+s)^2 + (1+s))/2 \Rightarrow 2 \leq s(1+s) \Rightarrow s > 0$

i.e., If $r > 2$, then both possibilities satisfy the inequality. □

5. Conclusion

In this study, new topological indices for hypergraphs were established and described, along with their significance for the outputs of graph operations. Additionally, modular structure construction was covered.

The hyperedge degree, $d_h(l)$, is a newly defined and described parameter that evaluates a hyperedge's connectedness to other hyperedges. Due to the vertices in a specific function l, the values of $d_h(l)$ show how correlated these hyperedges are, and $d_h(y)$ shows how many vertex y contained hyperedges in Z. By utilizing these hypergraph implementations in social networks such as Facebook, it becomes feasible to identify users who have greater influence over others. $d_h(y)$ value will be high for the users who are members of various groups. Membership in different groups is directly proportional to $d_h(y)$ value.

To circulate some important information (emergency issues) through Facebook or some other social media, this parameter can play a major role. In this situation, we can share this message in some groups having more $d_h(l)$. So with less time, without much effort, it will circulate and give much results. This work introduced one inequality that can be used to get a highly modular structure.

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Data availability : Not applicable

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