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ON DEFICIENCY POLYNOMIAL OF GRAPHS†

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ABSTRACT. The concept of graph polynomials, which is used to express various parameters of graphs, is one of the most trending areas of graph theory. In this article, we try to introduce a new graph polynomial may be called as deficiency polynomial. We determine the deficiency polynomial of certain standard classes of graphs and some other graphs obtained by graph theoretical operations. Further we introduce the concept of co-deficient graphs and trivial deficiency graphs and try to determine the graphs which are trivial deficient. The roots and stability of this polynomial is also dealt in detail.

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1. Introduction

Graph polynomials have been created for characterising graphs and assessing structural properties of systems using combinatorial graph parameters. Polynomials can be used to treat and solve a variety of problems in discrete mathematics in particular, graph theory in a very effective way. Graph polynomials, in general, encode the given graph's graph theoretical information in various ways. If a given graph polynomial encodes graph theoretic parameters, it might be of interest. The Wiener polynomial, whose coefficients are dependent on graph distances, is one example. Today, in the field of graph theory, there are plenty of graph polynomials namely vertex polynomial, edge polynomial, neighbourhood polynomial, chromatic polynomial, Hosoyo polynomial, Tutte polynomial, and so on [5].

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All graphs under our consideration are simple, connected and undirected. Let $G = (V(G), E(G))$ be a graph with order $n = |V(G)|$ and size $m = |E(G)|$. The **degree** of a vertex v_i in G is the number of edges incident on v_i and is denoted by $d(v_i)$, $deg_G(v)$ or simply d_i. The **distance** between two vertices u and v in G is the length of the shortest path joining them denoted by $d(u, v)$ in G. The **eccentricity** of v_i denoted by $e(v_i)$ or simply e_i is defined as $e(v_i) = max\{d(u, v_i) : u$ is a vertex of $G\}$. The n^{th} **power of a graph** G denoted as G^n has the same vertex set as that of G and two vertices u and v are adjacent in G^n, whenever $d(u, v) \leq n$ in G. The **join** of two graphs G_1 and G_2 denoted by $G_1 \vee G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$ together with all edges joining the vertices of V_1 to every vertex of V_2. We follow [1] for more terminologies and notations not mentioned here.

2. The Deficiency Polynomial of a Graph

In a graph, if every two distinct pair of vertices are joined by an edge, the graph is said to be **complete** [1]. Generally, a complete graph with n vertices is denoted by K_n. For K_n, degree of each vertex is $n-1$. Consider a graph G of order n, which is not complete. The vertex degree polynomial of G, $V_G(x)$ can be easily determined and the vertex degree polynomial of complete graph of the same order is nx^{n-1}. After adding a finite number of edges into G, the given graph G can be converted into K_n. Here we are defining a new polynomial called deficiency polynomial of G, which expresses the number of edges needed to be added into G so that G eventually becomes K_n. The number of edges to be added into G can be termed as the deficiency. Also we can see that under the operation $\widehat{\oplus}$ defined in this section, we get $D_G(x) \mathbin{\widehat{\oplus}} V_G(x) = nx^{n-1}$.

Definition 2.1. Let $G = (V, E)$ be a graph. Let $v_i, i = 1, 2, \ldots, n$ be the vertices of G with corresponding vertex degrees $d_i, i = 1, 2, \ldots, n$. The **deficiency polynomial** of the graph G, denoted by $D_G(x)$ is defined as

$$D_G(x) = \sum_{i=1}^{n} x^{\epsilon_i},$$

where $\epsilon_i = n - 1 - d_i, i = 1, 2, \ldots, n$.

Remark 2.1. The following are some immediate observations from the definition.

(1) The degree of deficiency polynomial of a graph is atmost $n-2$.
(2) The constant term in $D_G(x)$ represents the number of vertices in G with degree $n-1$.
(3) For a graph G with order n and size m, the general form of the deficiency polynomial of $D_G(x)$ is given by, $D_G(x) = a_0 + a_1x + a_2x^2 + \ldots +$

$a_{n-2}x^{n-2}$, where $a_i \in \mathbb{N}, 1 \leq a_i \leq n$ and $\sum\limits_{i=0}^{n-2} a_i = n$. Then

$$\frac{1}{2}\sum_{i=0}^{n-2} i\ a_i = \binom{n}{2} - m. \tag{1}$$

The right hand side of equation (1) gives the number of edges should be added to G so that G becomes a complete graph K_n.

Next, we try to introduce an operation on these polynomials.

Let G be a graph with order n. Consider the general form of vertex degree polynomial $V_G(x)$ and deficiency polynomial $D_G(x)$ of G

$$V_G(x) = a_0x^{n-1} + a_1x^{n-2} + a_2x^{n-3} + \ldots + a_{n-2}x \tag{2}$$

$$D_G(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-2}x^{n-2}, \tag{3}$$

where $a_i \in \mathbb{N}, 1 \leq a_i \leq n$ and $\sum\limits_{i=0}^{n-2} a_i = n$.

Consider the operation $\widehat{\oplus} : D_G(x) \times V_G(x) \longrightarrow V_{K_n}(x)$ on these polynomials;

$$\begin{aligned} D_G(x)\ \widehat{\oplus}\ V_G(x) &= \sum_{i=0}^{n-2} a_i x^{n-1-d_i}\ \widehat{\oplus}\ \sum_{i=0}^{n-2} a_i x^{d_i} \\ &= \sum_{i=0}^{n-2} a_i x^{n-1-d_i} \times x^{d_i} \\ &= \sum_{i=0}^{n-2} a_i x^{n-1} = nx^{n-1}, \end{aligned}$$

which is nothing but the vertex degree polynomial of the complete graph K_n. Clearly the operation is well-defined.

3. $D_G(x)$ of Some Standard Graphs

Proposition 3.1. *The following result holds for the corresponding class of graphs.*

(1) $D_{P_n}(x) = 2x^{n-2} + (n-2)x^{n-3}, n \geq 2$
(2) $D_{C_n}(x) = nx^{n-3}, n \geq 3$
(3) $D_{K_n}(x) = n, n \geq 1$
(4) $D_{K_{m,n}}(x) = mx^{m-1} + nx^{n-1}$
(5) $D_{B_{m,n}}(x) = x^n + x^m + (m+n)x^{m+n}$
(6) $D_{T_n}(x) = (n+1)x^{2n-4} + (n-2)x^{2n-6}, n \geq 2$
(7) $D_{W_n}(x) = nx^{n-3} + 1, n \geq 3.$
(8) $D_{L_{m,n}}(x) = x^{m+n-2} + x^{n-1} + (m-1)x^n + (n-1)x^{m+n-3}.$

Proof. (1) Let v_1, v_n be the pendant vertices and $v_2, v_3, \ldots, v_{n-1}$ be the internal vertices of P_n. Then $\epsilon_i = n-2$ for $i = 1, n$ and $\epsilon_i = n-3$, $2 \leq i \leq n-1$. This gives the result.

(2) For each vertex v_i in $C_n, d_i = 2, 1 \leq i \leq n$. This implies that $\epsilon_i = n-3$, $1 \leq i \leq n$.
(3) Degree of each vertex in K_n is $n-1$. Hence, $\epsilon_i = 0,\ 1 \leq i \leq n$.
(4) Let $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$ be the two partite sets. Then $deg(v_i) = n, 1 \leq i \leq m$ and $deg(u_j) = m, 1 \leq j \leq n$. Then,

$$\begin{aligned} D_{K_{m,n}}(x) &= \sum_{i=1}^{m} x^{m+n-1-n} + \sum_{j=1}^{n} x^{m+n-1-m} \\ &= mx^{m-1} + nx^{n-1}. \end{aligned}$$

(5) Let $B_{m,n}$ denotes the double star with $m+n+2$ vertices. Let v and u be the internal vertices and $v_i, 1 \leq i \leq m$ be the pendant vertices adjacent to the vertex v and $u_j, 1 \leq j \leq n$ be that of u. Then $deg(v) = m+1, deg(u) = n+1, deg(v_i) = deg(u_j) = 1, 1 \leq i \leq m$ and $1 \leq j \leq n$. Then

$$\begin{aligned} D_{B_{m,n}}(x) &= x^{m+n+1-(m+1)} + x^{m+n+1-(n+1)} + (m+n)x^{m+n} \\ &= x^n + x^m + (m+n)x^{m+n}. \end{aligned}$$

(6) A triangular snake T_n is obtained from the path P_n by replacing each edge of the path by a triangle C_3. T_n has $2n-1$ vertices. Let v_i be the vertices with degree 4 and u_j be the vertices with degree 2, where $1 \leq i \leq n-2$ and $1 \leq j \leq n+1$. This gives that, $\epsilon_i = 2n-6$ and $\epsilon_j = 2n-4$. These facts together with the definition of deficiency polynomial gives the result.
(7) Let v be the vertex of W_n with degree $n-1$ and $v_i, 1 \leq i \leq n-1$ be the vertices with degree 3. Then $\epsilon_i = n-4$ for the vertices $v_i, 1 \leq i \leq n-1$ and 0 for the vertex v.
(8) The lollipop graph $L_{m,n}$ is the graph obtained by joining a complete graph K_m to a path P_n with a bridge. Let $L_{m,n}$ denotes the lollipop graph. Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n where u_1 and u_n are pendant vertices and $v_1, v_2, \ldots, v_m$ be the vertices of K_m. A bridge is joined between v_1 and u_1. Then $deg(v_1) = m, deg(v_i) = m-1, 2 \leq i \leq m, deg(u_i) = 2, 1 \leq i \leq n-1, deg(u_n) = 1$. Then,

$$D_{L_{m,n}}(x) = x^{m+n-2} + x^{n-1} + (m-1)x^n + (n-1)x^{m+n-3}.$$

This completes the proof. □

The following are some immediate consequences of the above proposition

Corollary 3.2. *For $n \geq 3$,*

$$x\ D_{P_{n-1}}(x) = D_{P_n}(x) - x^{n-3}.$$

Proof. We know that $D_{P_n}(x) = 2x^{n-2} + (n-2)x^{n-3}$.
Consider,

$$\begin{aligned} D_{P_{n-1}}(x) &= 2x^{n-1-2} + (n-3)x^{n-1-3} \\ &= 2x^{n-3} + (n-3)x^{n-4} \end{aligned}$$

$$\begin{aligned}
x\ D_{P_{n-1}}(x) &= 2x^{n-3}x + (n-3)x^{n-4}x \\
&= 2x^{n-2} + (n-3)x^{n-3} \\
&= 2x^{n-2} + (n-2)x^{n-3} - (n-2)x^{n-3} + (n-3)x^{n-3} \\
&= D_{P_n}(x) + ((n-3)-(n-2))\, x^{n-3} \\
x\ D_{P_{n-1}}(x) &= D_{P_n}(x) - x^{n-3}.
\end{aligned}$$

This completes the proof. □

Corollary 3.3. *For $n \geq 3$,*

$$D_{C_{n-1}}(x) = \frac{n-1}{nx} D_{C_n}(x).$$

Proof. We know that $D_{C_n}(x) = nx^{n-3}$.
Consider,

$$\begin{aligned}
D_{C_{n-1}}(x) &= (n-1)x^{n-4} \\
&= nx^{n-4} - x^{n-4} \\
&= D_{C_n}(x)x^{-1} - \frac{1}{n} D_{C_n}(x)x^{-1} \\
&= D_{C_n}(x)\left(\frac{1}{x} - \frac{1}{nx}\right) \\
&= \frac{n-1}{nx} D_{C_n}(x).
\end{aligned}$$

This completes the proof. □

In the following result, deficiency polynomial of $K_{m,n}$ is written in terms of deficiency polynomials of two cycles.

Corollary 3.4. *For $m, n \geq 3$,*

$$D_{K_{m,n}}(x) = \frac{x^2}{mn}\left(nD_{C_m}(x) + mD_{C_n}(x)\right).$$

Proof. We know that $D_{K_{m,n}}(x) = mx^{m-1} + nx^{n-1}$.
Consider,

$$\begin{aligned}
D_{K_{m,n}}(x) &= mx^{m-1} + nx^{n-1} \\
&= \frac{x^2}{m} D_{C_m}(x) + \frac{x^2}{n} D_{C_n}(x) \\
&= x^2\left(\frac{D_{C_m}(x)}{m} + \frac{D_{C_n}(x)}{n}\right) \\
&= \frac{x^2}{mn}\left(n\ D_{C_m}(x) + m\ D_{C_n}(x)\right).
\end{aligned}$$

This completes the proof. □

Proposition 3.5. *For an r-regular graph G on n vertices $D_G(x) = nx^{n-r-1}$.*

Proof. For each vertex $v_i, d_i = r, 1 \leq i \leq n$. This implies that $\epsilon_i = n - 1 - r$, $1 \leq i \leq n$. This completes the proof. □

4. Deficiency Polynomial of Graphs obtained from Some Graph Operations

In this section we compute deficiency polynomial of certain graphs which are obtained by performing some graph theoretic operations such as join, corona product, cartesian product. Further we construct some graphs say shadow graphs and splitting graphs and get their deficiency polynomial.

Theorem 4.1. *Let G and H be two graphs, $G \vee H$ be their join. Then*

$$D_{G\vee H}(x) = D_G(x) + D_H(x).$$

Proof. Let G and H be two graphs with number of vertices m, n respectively, $m, n \geq 2$. Let $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ be the vertices of G and H with $deg_G(v_i) = d_i$ and $deg_H(u_j) = d_j^*, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The deficiency polynomial of G and H is given respectively by $D_G(x) = \sum\limits_{i=1}^{m} x^{m-1-d_i}$, $D_H(x) = \sum\limits_{j=1}^{n} x^{n-1-d_j^*}$.

Consider, the join $G \vee H$ of the graphs G and H. $G \vee H$ has the same vertex set $v_i, u_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Here $deg_{G\vee H}(v_i) = d_i + n$ and $deg_{G\vee H}(u_j) = d_j^* + m, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.
Consider,

$$\begin{aligned} D_{G\vee H}(x) &= \sum_{i=1}^{m} x^{m+n-1-(d_i+n)} + \sum_{j=1}^{n} x^{m+n-1-(d_j^*+m)} \\ &= \sum_{i=1}^{m} x^{m-1-d_i} + \sum_{j=1}^{n} x^{n-1-d_j^*} \\ &= D_G(x) + D_H(x). \end{aligned}$$

This completes the proof. □

Theorem 4.2. *Let G and H be two graphs, the corona product of G and H be denoted as $G \odot H$. Then*

$$D_{G\odot H}(x) = x^{n(m-1)} D_G(x) + m x^{(n+1)(m-1)} D_H(x).$$

Proof. Let $v_i, i = 1, 2, \ldots, m$ and $u_j, j = 1, 2, \ldots, n$ be the vertices of G and H respectively. Let $deg_G(v_i) = d_i, i = 1, 2, \ldots, m$ and $deg_H(u_j) = d_j^*, j = 1, 2, \ldots, n$. Consider the corona product $G \odot H$ of G and H. Then $G \odot H$ has $m(n+1)$ vertices. The vertices of $G \odot H$ is relabeled as follows: the vertices of H which are adjacent to v_i in $G \odot H$ is labeled as $u_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

Then $deg_{G\odot H}(v_i) = d_i + n$ and $deg_{G\odot H}(u_{ij}) = d_j^* + 1$.
Consider,

$$D_{G\odot H}(x) = \sum_{i=1}^{m} x^{m(n+1)-1-(d_i+n)} + \sum_{j=1}^{n} x^{m(n+1)-1-(d_j^*+1)}$$
$$= x^{n(m-1)} D_G(x) + m x^{(n+1)(m-1)} D_H(x).$$

Hence the theorem. □

Theorem 4.3. *For any two graphs G_1 and G_2 with order $m, n \geq 2$, the deficiency polynomial of the cartesian product of G_1 and G_2 is given by*

$$D_{G_1\times G_2}(x) = x^{(m-1)(n-1)} D_{G_1}(x) D_{G_2}(x).$$

Proof. Let $u_i \in V(G_1)$, with $deg(u_i) = d_i, i = 1, 2, \ldots, m$ and $v_j \in V(G_2)$, with $deg(v_j) = d_j^*, j = 1, 2, \ldots, n$. Consider the cartesian product $G_1 \times G_2$. Let $(u_i, v_j) \in V(G_1 \times G_2)$. Then $deg(u_i, v_j) = deg_{G_1}(u_i) + deg_{G_2}(v_j) = d_i + d_j^*$.
Consider,

$$D_{G_1\times G_2}(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} x^{mn-1-(d_i+d_j^*)}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} x^{mn-1-(m-1-d_i)+m-1-(n-1-d_j^*)+n-1}$$
$$= x^{(m-1)(n-1)} D_{G_1}(x) D_{G_2}(x).$$

This completes the proof. □

Corollary 4.4. *For the planar grids $m, n \geq 2$,*

$$D_{P_m\times P_n}(x) = 4x^{mn-3} + [2(m+n)-8]\, x^{mn-4} + [mn - 2(m+n) + 4]\, x^{mn-5}.$$

Corollary 4.5. *For the ladder graph $L_n, n \geq 2$*

$$D_{L_n}(x) = 4x^{2n-3} + 2(n-2)x^{2n-4}.$$

Corollary 4.6. *For the torus grids $m, n \geq 3$,*

$$D_{C_m\times C_n}(x) = mnx^{mn-5}.$$

Corollary 4.7. *For $m, n \geq 1$,*

$$D_{K_m\times K_n}(x) = mnx^{(m-1)(n-1)}.$$

Corollary 4.8. *For the prisms $m \geq 3, n \geq 2$,*

$$D_{C_m\times P_n}(x) = 2mx^{mn-4} + m(n-2)x^{mn-5}.$$

The splitting graph $S'(G)$ of a graph G is obtained by adding to each vertex v a new vertex v', such that v' is adjacent to every vertex that is adjacent to v in G [4].

Theorem 4.9. *For the splitting graph of G,*

$$D_{S'(G)}(x) = x\sum_{i=1}^{n} x^{2\epsilon_i} + x^n D_G(x)$$

where $\epsilon_i = n-1-d_i, i = 1,2,\ldots,n$.

Proof. Let v_i be the vertices of G with $deg_G(v_i) = d_i, i = 1,2,\ldots,n$. Let v_i' be the corresponding vertex of v_i in $S'(G)$. Clearly $S'(G)$ has $2n$ vertices. Then $deg_{S'(G)}(v_i) = 2d_i$ and $deg_{S'(G)}(v_i') = d_i \ \forall\ i = 1,2,\ldots n$.
Consider,

$$\begin{aligned} D_{S'(G)}(x) &= \sum_{i=1}^{n} x^{2n-1-2d_i} + \sum_{i=1}^{n} x^{2n-1-d_i} \\ &= x\sum_{i=1}^{n} \left(x^{n-1-d_i}\right)^2 + x^n \sum_{i=1}^{n} x^{n-1-d_i} \\ &= x\sum_{i=1}^{n} x^{2\epsilon_i} + x^n D_G(x). \end{aligned}$$

This completes the proof. □

The shadow graph $S_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G'' [4].

Theorem 4.10. *For the shadow graph of G,*

$$D_{S_2(G)}(x) = x\sum_{i=1}^{n} x^{2\epsilon_i},$$

where $\epsilon_i = n-1-d_i, i = 1,2,\ldots,n$.

5. Co-Deficient Graphs and Deficiency Equivalent Class of $D_G(x)$

In this section, deficiency polynomial of graphs which are isomorphic and not isomorphic to each other are discussed.

Theorem 5.1. *If G and H are two isomorphic graphs, then $D_G(x) = D_H(x)$.*

Proof. Trivially holds. □

The converse of the above theorem need not be true. That is, even if G and H have the same deficiency polynomial, they need not be isomorphic to each other.

For example, consider figure 1.

Here both the graphs have the same deficiency polynomial, but they are not isomorphic. In this case, we can define the following.

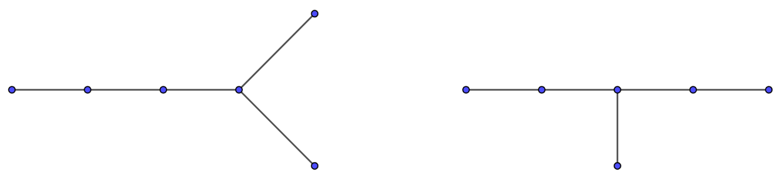

FIGURE 1. Non-isomorphic graphs with same deficiency polynomials

Definition 5.2. Two non-isomorphic graphs G and H are said to be **co-deficient graphs** (written $G \overset{D}{\sim} H$) if $D_G(x) = D_H(x)$.

Definition 5.3. Let $G_1, G_2, \ldots, G_n$ be the graphs having the same deficiency polynomial and $G_i \ncong G_j \ \forall\ i \neq j, i, j \in \{1, 2, \ldots, n\}$.
Then $\mathcal{C} = \{G_i, \forall\ i = 1, 2, \ldots, n\}$ is called a deficiency equivalent class. A graph G is said to be **trivial deficiency**, if there exists no H ($H \ncong G$) with the deficiency polynomial as G.

For example $K_n,\ n \geq 2$ and $C_n,\ n \geq 3$ are trivial deficiency graphs.

Although each graph has a unique degree sequence, two non-isomorphic graphs may have **identical degree sequence**[1].

Theorem 5.4. *Two non-isomorphic graphs with identical degree sequence must be co-deficient.*

Theorem 5.5. *Let G be a trivial deficiency graph. Then $G \vee K_n$ is also a trivial deficiency graph.*

Proof. Let $D_G(x)$ be the deficiency polynomial of G. Since G is a trivial deficiency graph, no other graph has the deficiency polynomial as that of G. Also we have,

$$\begin{aligned} D_{G \vee K_n}(x) &= D_G(x) + D_{K_n}(x) \\ &= D_G(x) + n. \end{aligned}$$

Hence $G \vee K_n$ is also a trivial deficiency graph. □

Let G and H be co-deficient graphs. That is $G \ncong H$ with $D_G(x) = D_H(x)$. Naturally a question arises: "How can one generate infinitely many co-deficient graphs from the given pair of co-deficient graphs G and H?" The following result gives an affirmative answer to this question.

Theorem 5.6. *Let G and H be two graphs such that $G \overset{D}{\sim} H$, then $S(G) \overset{D}{\sim} S(H)$.*

Proof. Let the vertex set of G and H be v_i and u_j, with $deg_G(v_i) = d_i$ and $deg_H(u_j) = d_j^*$ where $i, j = 1, 2, \ldots, n$ with size m. Since $G \overset{D}{\sim} H$, we have

$D_G(x) = D_H(x)$. Let $S(G)$ and $S(H)$ be the subdivision graph of G and H respectively. Let w_i and w_i^* be the newly introduced vertices of G and H respectively after the process subdivision, where $i = 1, 2, \ldots, m$. Then, $deg_{S(G)}(v_i) = d_i, deg_{S(H)}(v_i) = d_i^*$ and $deg_{S(G)}(w_i) = deg_{S(H)}(w_i^*) = 2$.

Consider,

$$\begin{aligned} D_{S(G)}(x) &= \sum_{i=1}^{n} x^{m+n-1-d_i} + m \times x^{m+n-1-2} \\ &= x^m D_G(x) + m x^{m+n-3}. \end{aligned}$$

Similarly, $D_{S(H)}(x) = x^m D_H(x) + m x^{m+n-3}$.
This completes the proof. □

An illustration of the theorem is given below.

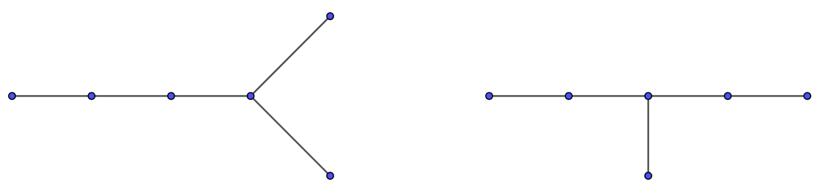

FIGURE 2. Co-deficient graphs with deficiency polynomial $x^2 + 2x^3 + 3x^4$

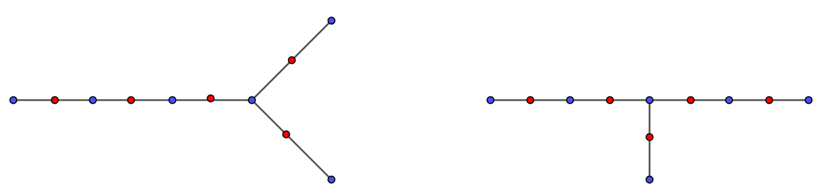

FIGURE 3. Co-deficient graphs obtained from the graphs in figure 2 after subdivision

6. Roots and Stability Analysis of Deficiency Polynomial

The solutions of the equation $D_G(x) = 0$ is called the roots of deficiency polynomial of the given graph. Since all the coefficients of $D_G(x)$ is positive, it can't have a positive root. So we can say that $(0, \infty)$ is a zero-free interval of the deficiency polynomial $D_G(x)$ of any graph G.

Next we give a characterization for the deficiency polynomial of a graph G to have 0 as a root.

Theorem 6.1. *Let G be a graph of order n, then 0 is a root of $D_G(x)$ if and only if G has no vertex of degree $n-1$. Moreover the multiplicity of 0 is $n-1-\Delta$, where Δ is the maximum degree of the graph.*

Proof. Assume that 0 is a root of $D_G(x)$. So, x is a factor of $D_G(x)$. Hence by division algorithm, we can find a polynomial $P(x)$ such that $D_G(x) = x\ P(x)$, where $deg(P(x)) < deg(D_G(x))$. That is $x\ P(x)$ is a polynomial having no constant terms. This implies that $D_G(x)$ has no constant term. From remark 2.1, we know that the constant term in $D_G(x)$ represents the number of vertices in G with degree $n-1$. In this case we have shown that, $D_G(x)$ has no constant terms, which shows that G has no vertex of degree $n-1$.

Conversely assume that $D_G(x)$ has no vertex of degree $n-1$. Consider

$$\begin{aligned} D_G(x) &= a_1x + a_2x^2 + \ldots + a_{n-2}x^{n-2} \\ &= x(a_1 + a_2x + \ldots + a_{n-2}x^{n-3}) \end{aligned}$$

where $\sum\limits_{i=1}^{n-2} a_i = n$. $D_G(x) = 0$ implies $x = 0$ is a root. This completes the proof. □

Proposition 6.2. *For an r-regular graph on n vertices other than K_n, 0 is the only root with multiplicity $n-r-1$.*

In the next section we discuss about the stability of deficiency polynomial of certain graphs.

Definition 6.3. [2] A polynomial $f(z_1, \ldots, z_n)$ is said to be **stable** with respect to a region $\Omega \subseteq \mathrm{C}^n$ if no root of f lies in Ω.

Specifically, Hurwitz polynomials and Schur polynomials are polynomials that exhibit stability with regard to the open left half plane and the open unit disc, respectively.

Next we try to determine the stability of deficiency polynomial of certain classes of graphs with respect to the open right half plane.

Theorem 6.4. *Let P_n be the path with $n \geq 2$ vertices, then $D_{P_n}(x)$ is stable.*

Proof. We know that for each $n \geq 2$, $D_{P_n}(x) = 2x^{n-2} + (n-2)x^{n-3}$. To find the roots of $D_{P_n}(x)$ consider the following.

$$D_{P_n}(x) = 2x^{n-2} + (n-2)x^{n-3} = 0$$

$$x^{n-3}\left[2x+n-2\right]=0.$$

This gives the roots of $D_{P_n}(x)$ as $x=0$ (Since 0 repeats $n-3$ times as a root of $D_{P_n}(x)$, 0 has multiplicity $n-3$), $\frac{-(n-2)}{2}$. Clearly $x=\frac{-(n-2)}{2}$ lies in the left half plane. Therefore $D_{P_n}(x)$ is stable. This completes the proof. □

Theorem 6.5. *Deficiency polynomial of every r-regular graph is stable, if there exists roots.*

Remark 6.1. For the complete graph K_n no roots exists.

Proof. Since 0 is the only root of an r-regular graph and it does not belongs to the open right half plane, the result holds. □

Theorem 6.6. *The deficiency polynomial of cartesian product of G_1 and G_2 is stable if both the deficiency polynomial of G_1 and G_2 are stable.*

Proof. Let G_1 and G_2 be two graphs with order m and n,$m,n\geq 2$ respectively. Then we have to show that $D_{G_1\times G_2}(x)$ is stable if $D_{G_1}(x)$ and $D_{G_2}(x)$ are stable. From theorem 4.3, the deficiency polynomial of cartesian product of G_1 and G_2 is given by $D_{G_1\times G_2}(x)=x^{(m-1)(n-1)}D_{G_1}(x)D_{G_2}(x)$. To find the roots of $D_{G_1\times G_2}(x)$, consider

$$x^{(m-1)(n-1)}D_{G_1}(x)D_{G_2}(x)=0.$$

This gives the roots as $x=0$ (Since 0 repeats $mn-m-n+1$ times as a root of $D_{G_1\times G_2}(x)$, 0 has multiplicity $mn-m-n+1$) together with the roots of $D_{G_1}(x)$ and $D_{G_2}(x)$. Since $x=0$ does not belong to the open right half plane, the stability of $D_{G_1\times G_2}(x)$ is completely determined by the roots of $D_{G_1}(x)$ and $D_{G_2}(x)$. Hence the theorem. □

From theorem 6.6, it is clear that the deficiency roots of $G_1\times G_2$ is exactly that of G_1 and G_2 together with multiplicity.
Following are some corollaries of theorem 6.6.

Corollary 6.7. *For $m,n\geq 2, D_{P_m\times P_n}(x)$ is stable.*

Proof. From corollary 4.4, the deficiency polynomial of $P_m\times P_n$ is given by $D_{P_m\times P_n}(x)=4x^{mn-3}+\left[2(m+n)-8\right]x^{mn-4}+\left[mn-2(m+n)+4\right]x^{mn-5}$. On solving $D_{P_m\times P_n}(x)=0$ yields, $x=0$ (with multiplicity $mn-5$), $x=1-\frac{n}{2}$ and $x=1-\frac{m}{2}$. Clearly for $m,n\geq 2$, $x=1-\frac{n}{2}$ and $x=1-\frac{m}{2}$ are negative. This shows that no roots of $P_m\times P_n$ lies in the open right half plane. This completes the proof. □

In a similar manner the following can also be proved.

Corollary 6.8. *For $n\geq 2$, the deficiency polynomial of Ladder graph L_n is stable.*

Corollary 6.9. *For* $m, n \geq 1, D_{K_m \times K_n}(x)$ *is stable.*

Proof. Since $K_m \times K_n$ is a regular graph with degree $m+n-2$, using theorem 6.5 the proof holds. □

Corollary 6.10. *For* $m, n \geq 3, D_{C_m \times C_n}(x)$ *is stable.*

Corollary 6.11. *For* $m \geq 3, n \geq 2,$ $D_{C_m \times P_n}(x)$ *is stable.*

Theorem 6.12. *For the wheel graph,* $D_{W_n}(x)$ *is stable for* $n = 4, 5$.

Proof. From theorem 3.1 (7), the deficiency polynomial of W_n is given by $D_{W_n}(x) = nx^{n-3} + 1, n \geq 3$.
Case 1: $n = 4$
In this case $D_{W_4}(x) = 4x + 1$. On solving $D_{W_4}(x) = 0$ yields $x = \frac{-1}{4}$, which belong to the left half plane. This shows that $D_{W_4}(x)$ is stable.
Case 2: $n = 5$
In this case $D_{W_5}(x) = 5x^2 + 1$. To determine the roots of $D_{W_5}(x)$, consider $D_{W_5}(x) = 0$. This gives $x = \pm\frac{i}{\sqrt{5}}$, which lies in the imaginary axis. This shows that $D_{W_5}(x)$ is stable.
From the above two cases the result holds. □

Remark 6.2. For $n \geq 6$, $D_{W_n}(x)$ need not be stable. For $n = 6, 7, 8$ and 9, the GeoGebra plots of $D_{W_n}(x)$ are given below:

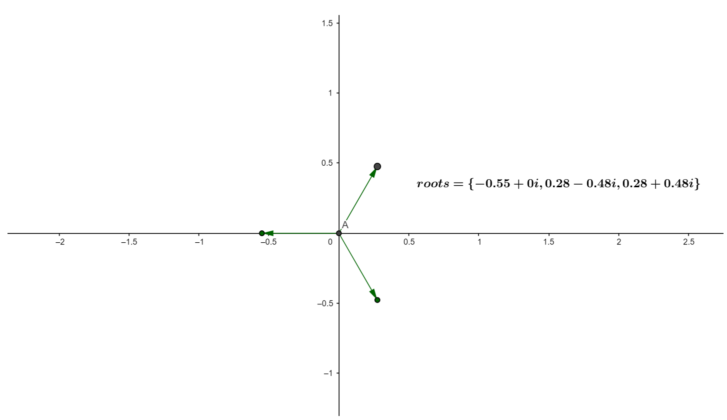

FIGURE 4. $n = 6$

Theorem 6.13. [3] *Let* $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, *where* $a_i \in \mathbb{C}$. *Then, inside the circle* $|z| = 1 + \max_i |a_i|$, *there are exactly* n *roots of* f *(multiplicities counted).*

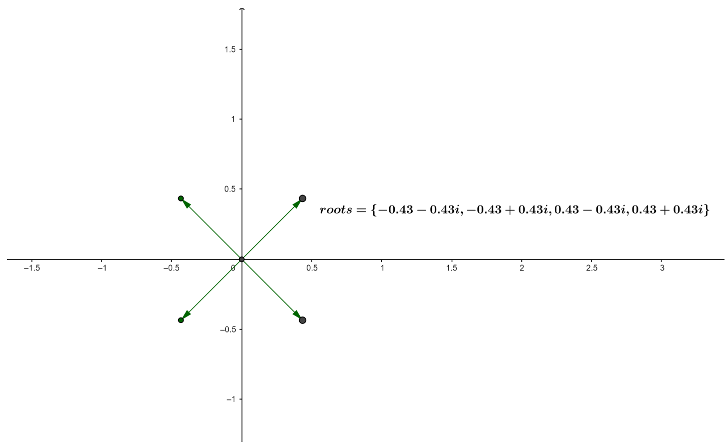

FIGURE 5. $n = 7$

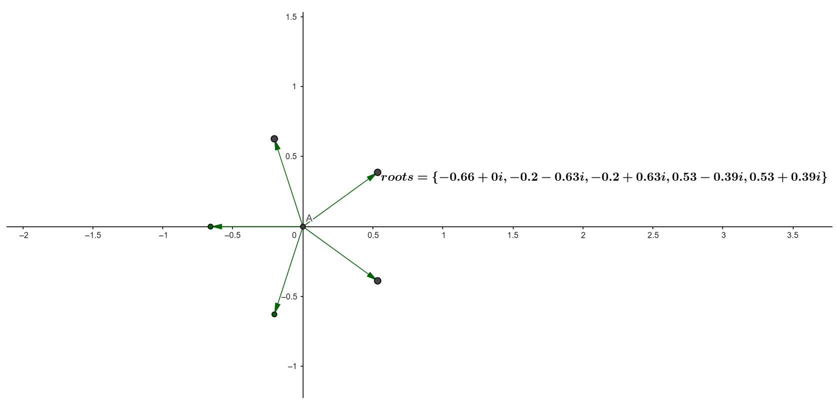

FIGURE 6. $n = 8$

Theorem 6.14. *For the lollipop graph $L_{m,n}$, $m \geq 3, n \geq 1$, all the roots of $D_{L_{m,n}}(x)$ lies inside the circle $|z| = \max\{m, n\}$.*

Proof. From proposition 3.1(8) we have,

$$D_{L_{m,n}}(x) = x^{m+n-2} + x^{n-1} + (m-1)x^n + (n-1)x^{m+n-3}.$$

Case 1: if m is maximum
In this case using theorem 6.13, $|a_i| = \max\{m-1, 1, n-1\} = m-1$. This shows that the radius of the circle is $1 + m - 1 = m$. This proves the case.
Case 2: if n is maximum
As in case 1, $|a_i| = \max\{m-1, 1, n-1\} = n-1$. This shows that the radius of the circle is $1 + n - 1 = n$. This proves the case.
Hence the result. $\square$

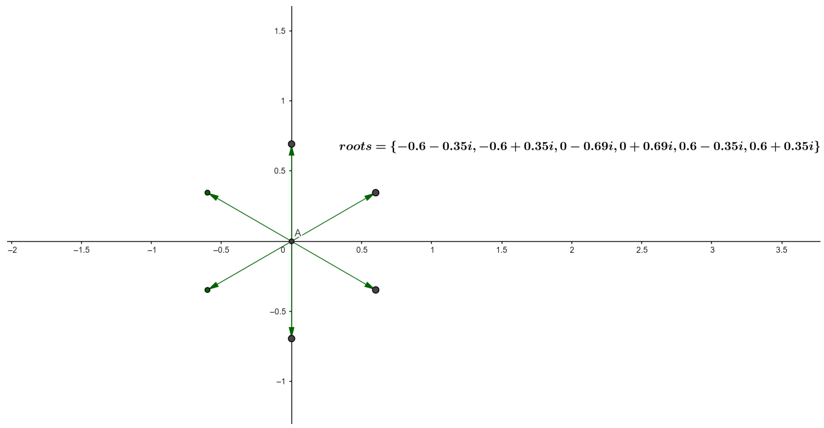

FIGURE 7. $n = 9$

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